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Analytic Solutions to the Darcy-Lapwood-Brinkman Equation with Variable Permeability

S.O. Alharbi[1], T.L. Alderson[2], M.H. Hamdan[3]
[1]*(Department Of Mathematical Sciences, University Of New Brunswick, Saint John, N.B., Canada, E2L 4L5 On Leave From Majmaah University, Kingdom Of Saudi Arabia.)*
[2,3] *(Department Of Mathematical Sciences, University Of New Brunswick, Saint John, N.B., Canada, E2L 4L5)*

ABSTRACT
Three exact solutions to the Darcy-Lapwood-Brinkman equation with variable permeability are obtained in this work. Solutions are obtained for a given vorticity distribution, taken as a function of the streamfunction. Classification of the flow field is provided and comparison is made with the solutions obtained when permeability is constant. Interdependence of Reynolds number and variable permeability is emphasized.
Keywords - Darcy-Laprood-Brinkman equation, Variable permeability, Exact solutions

I. INTRODUCTION

Flow through variable permeability porous media finds applications in oil and gas recovery, industrial and biomechanical processes and in natural environmental settings and agriculture, [1]. Naturally occurring media are of variable porosity and permeability, and the flow through which is governed by flow models with permeability tensor, [1], [2]. In some idealization of heterogeneous and inhomogeneous media, and in two-dimensional flow simulations, the permeability can be taken as a variable function of one or two independent variables [2]. A number of studies have implemented this approach (cf. [2], [3], [4], and the references therein). Variable permeability simulation has also proved to be indispensable in the study of the transition layer, [4], (defined here as a thin layer that is sandwiched between a constant permeability porous layer and a free-space channel, and the flow through which is governed by Brinkman's equation).

Models of flow through porous media come in a variety of forms depending on whether viscous shear effects and inertial effects are important. In the presence of solid boundaries, shear effects are important and it has been customary to use Brinkman's equation to model the flow, [1]. When, in addition, inertial effects are important one resorts to the Darcy-Lapwood-Brinkman (DLB) equation (discussed in the current work), which takes into account macroscopic inertial effects and viscous shear effects. If micro-inertial effects are important, one resorts to a Forchheimer-type inertial model, [1].

The DLB equation, discussed in section 2 of this work, resembles the Navier-Stokes equations, and involves a viscous damping (Darcy-like) term. Not unlike the Navier-Stokes equations, exact solutions are rare due to the nonlinearity of the equations and the inapplicability of the superposition Principle to nonlinear partial differential equations, (cf. [5], [6], [7], [8], [9], [10]). Taylor [11] identified the source of nonlinearity as the convective inertial terms, which vanish in two-dimensional flows when the vorticity of the flow is a function of the streamfunction of the flow.

By taking the vorticity to be proportional to the streamfunction of the flow, Taylor's solution [11] represents a double infinite array of vortices decaying exponentially with time. Kovasznay, [12], extended Taylor's approach and linearised the Navier–Stokes equations by taking the vorticity to be proportional to the streamfunction perturbed by a uniform stream. Kovasznay's two-dimensional solution represents the flow behind (downstream of) a two-dimensional grid. Two solutions representing the reverse flow over a flat plate with suction and blowing were obtained by Lin and Tobak [13], who extended Kovasznay's approach. Various other authors have obtained exact solutions to the Navier–Stokes and other equations for special types of flow (cf. [14], [15], [16], and the reviews in [9], and [10]).

Most methods used in linearizing the Navier-Stokes equations have been used in the analysis of the DLB equation with constant permeability. The case of variable permeability is treated in this work, where we consider two-dimensional flow through a porous medium governed by a variable-permeability DLB equation and find three analytic solutions for a prescribed permeability function of one space variable, when the vorticity of the flow is a function of the streamfunction of the flow. We take the variable permeability to be a function of position and a function of Reynolds number, since in the case of flow with constant permeability, Reynolds number and permeability are interdependent. The current study may prove to be of importance in stability studies of flow through variable permeability periodic porous structures and in the study of flows that deviate from base flows in porous structures.

II. GOVERNING EQUATIONS

The steady flow of an incompressible fluid through a porous medium composed of a mush zone is governed by the equations of continuity and momentum, written respectively as, [1]

$$\nabla \bullet \vec{v} = 0 \quad \ldots(1)$$

$$\rho(\vec{v} \bullet \nabla)\vec{v} = -\nabla p + \mu^* \nabla^2 \vec{v} - \frac{\mu}{k}\vec{v} \quad \ldots(2)$$

where $\vec{v}$ is the velocity vector field, p is the pressure, ρ is the fluid density, μ is the viscosity of the base fluid, μ^* is the effective viscosity of the fluid as it occupies the porous medium, k is the permeability (considered here a scalar function of position), ∇ is the gradient operator and ∇^2 is the laplacian. In the absence of definite information about the relationship between μ^* and μ, we will take $\mu^* = \mu$ in this work.

Considering the flow in two space dimensions, *x* and *y*, we take $\vec{v} = (u, v)$, $k = k(x, y)$ and $p = p(x, y)$. Using the dimensionless quantities defined by

$$(x^*, y^*) = \frac{(x, y)}{L}; (u^*, v^*) = \frac{(u, v)}{U}; k^* = \frac{k}{L^2}; p^* = \frac{p}{\rho U^2} \quad \ldots(3)$$

where *L* is a characteristic length and *U* a characteristic velocity, equation (1) then takes the following dimensionless form after dropping the asterisk (*):

$$u_x + v_y = 0 \quad \ldots(4)$$

And momentum equations (2) are expressed in the following dimensionless form:

$$\mathrm{Re}(uu_x + vu_y) = -P_x + \nabla^2 u - \frac{u}{k} \quad \ldots(5)$$

$$\mathrm{Re}(uv_x + vv_y) = -P_y + \nabla^2 v - \frac{v}{k} \quad \ldots(6)$$

where $\mathrm{Re} = \frac{\rho UL}{\mu}$ is the Reynolds number.

System (4), (5), and (6) can be conveniently written in streamfunction-vorticity form as follows. Equation (4) implies the existence of a dimensionless streamfunction $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y} \quad \ldots(7)$$

and

$$v = -\frac{\partial \psi}{\partial x}. \quad \ldots(8)$$

Dimensionless vorticity, ξ, in two dimensions is defined as:

$$\xi = \nabla \times \vec{v} = v_x - u_y. \quad \ldots(9)$$

Using (7) and (8) in (9), we obtain the streamfunction equation

$$\xi = -\psi_{xx} - \psi_{yy} = -\nabla^2 \psi. \quad \ldots(10)$$

Vorticity equation is obtained from equations (5) and (6) by eliminating the pressure term through differentiation, and can be written in the following equivalent forms

$$\mathrm{Re}[\psi_y \xi_x - \psi_x \xi_y] = \nabla^2 \xi - \frac{1}{k}\xi - \frac{k_x \psi_x}{k^2} - \frac{k_y \psi_y}{k^2} \quad \ldots(11)$$

$$\mathrm{Re}[u\xi_x + v\xi_y] = \nabla^2 \xi - \frac{1}{k}\xi + v\frac{k_x}{k^2} - u\frac{k_y}{k^2}. \quad \ldots(12)$$

III. SOLUTION METHODOLOGY

In order to solve equations (10) and (11 (or (12)) for ψ and ξ, we assume vorticity to be a function of the streamfunction defined by

$$\xi = \frac{\alpha y}{\mathrm{Re}} - \alpha \psi \quad \ldots(13)$$

Where α is a parameter to be determined.

Since equations (10) and (11) or (12) represent two equations in the two unknowns ψ and ξ, we must assume the form of the permeability function, $k(x, y)$. In the current work, we assume k to be a function of x only or a function of y only. Since the vorticity in (13) involves y explicitly, we will assume that

$$k = k(x) = \frac{\alpha \,\mathrm{Re}}{x}. \quad \ldots(14)$$

Equation (14) is based on the assumption that the model equations (Darcy-Lapwood-Brinkman equation (2)) is valid when inertial effects are significant, hence $\mathrm{Re} > 0$. It is clear that when permeability increases, the flow is faster, and *Re* increases, and conversely. This choice of permeability function is characteristic of flow in a porous domain where permeability decreases downstream as x increases. We also assume here that

α is a permeability-adjustment parameter in the sense that it is a parameter that depends on the local value of permeability along a given line $x =$ constant.

If $\mathrm{Re} = 0$, then the Darcy-Lapwood-Brinkman model, equation (2), reduces to the inertia-free Brinkman's equation, which warrants different choice of permeability function.

Now, substituting (13) and (14) in (11), and simplifying, we obtain

$$\psi_x + \frac{\alpha(x-\alpha^2 \mathrm{Re})}{\alpha^2 \mathrm{Re}+1}\psi = \frac{\alpha}{\alpha^2 \mathrm{Re}+1}\left[\frac{x-\alpha^2 \mathrm{Re}}{\mathrm{Re}}\right]y \;.$$
…(15)

Equation (15) has the integration factor given by

$$I.F. = \exp\left[\frac{\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(16)

and solution given by

$$\psi = \frac{y}{\mathrm{Re}} + f(y)\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(17)

Where $f(y)$ is an arbitrary function of y.

Using (10), (13) and (17), we obtain the following equation that must be satisfied by $f(y)$

$$f''(y) - \{\alpha + \frac{\alpha}{\alpha^2 \mathrm{Re}+1} - \frac{\alpha^2}{(\alpha^2 \mathrm{Re}+1)^2}[x^2 - 2\alpha^2 \mathrm{Re}\;x + \alpha^4 \mathrm{Re}^2]\} f(y) = 0$$
. …(18)

Equating coefficients of *x* to power, we obtain:

Coefficient of x^2:

$$-\frac{\alpha^2}{(\alpha^2 \mathrm{Re}+1)^2} f(y) = 0$$
…(19)

Coefficient of x:

$$\frac{2\alpha^4 \mathrm{Re}}{(\alpha^2 \mathrm{Re}+1)^2} f(y) = 0$$
…(20)

Coefficient of x^0:

$$f''(y) - \{\alpha + \frac{\alpha}{\alpha^2 \mathrm{Re}+1} - \frac{\alpha^6 \mathrm{Re}^2}{(\alpha^2 \mathrm{Re}+1)^2}\} f(y) = 0 \;.$$
…(21)

Equations (19) and (20) yield $f(y) = 0$ when $\alpha \neq 0$ and $\mathrm{Re} > 0$. Using $f(y) = 0$ in (17) gives $\psi = \frac{y}{\mathrm{Re}}$, with horizontal streamlines $\psi =$ constant, and velocity components $v = 0$, and $u = \frac{1}{\mathrm{Re}}$ which is a decreasing horizontal velocity with increasing Reynolds number.

If $f(y) \neq 0$, then by letting

$$\beta = \alpha + \frac{\alpha}{\alpha^2 \mathrm{Re}+1} - \frac{\alpha^6 \mathrm{Re}^2}{(\alpha^2 \mathrm{Re}+1)^2}$$
…(22)

equation (21) takes the form

$$f''(y) - \beta f(y) = 0 \;.$$
…(23)

Auxiliary equation of (23) is:

$$m^2 - \beta = 0$$
…(24)

with characteristic roots

$$m = \mp\sqrt{\beta} \;.$$
…(25)

Three cases arise depending on the value of β:

Case 1: $\beta > 0$

Solution to (23) takes the form

$$f(y) = c_1 e^{\sqrt{\beta}\,y} + c_2 e^{-\sqrt{\beta}\,y}$$
…(26)

where c_1 and c_2 are arbitrary constants. The streamfunction, solution (17) thus becomes

$$\psi = \frac{y}{\mathrm{Re}} + [c_1 e^{\sqrt{\beta}\,y} + c_2 e^{-\sqrt{\beta}\,y}]\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(27)

with velocity components given by

$$u = \psi_y = \frac{1}{\mathrm{Re}} + \sqrt{\beta}[c_1 e^{\sqrt{\beta}\,y} - c_2 e^{-\sqrt{\beta}\,y}]\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(28)

$$v = -\psi_x = -[c_1 e^{\sqrt{\beta}\,y} + c_2 e^{-\sqrt{\beta}\,y}]\left[\frac{-\alpha(x-\alpha^2 \mathrm{Re})}{\alpha^2 \mathrm{Re}+1}\right]\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(29)

and Vorticity takes the form

$$\xi = \frac{\alpha y}{\mathrm{Re}} - \alpha\psi = -\alpha[c_1 e^{\sqrt{\beta}\,y} + c_2 e^{-\sqrt{\beta}\,y}]\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
. …(30)

Case 2: $\beta = 0$

If $\beta = 0$, then equation (26) is replaced by

$$f(y) = c_1 + c_2 y \;.$$
…(31)

The streamfunction, solution (27) is thus replaced by

$$\psi = \frac{y}{\mathrm{Re}} + [c_1 + c_2 y]\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(32)

with velocity components given by

$$u = \psi_y = \frac{1}{\mathrm{Re}} + c_2 \exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(33)

$$v = -\psi_x = -[c_1 + c_2 y]\left[\frac{-\alpha(x-\alpha^2 \mathrm{Re})}{\alpha^2 \mathrm{Re}+1}\right]\exp\left[\frac{-\alpha(x^2/2-\alpha^2 \mathrm{Re}\;x)}{\alpha^2 \mathrm{Re}+1}\right]$$
…(34)

and Vorticity takes the form

$$\xi = \frac{\alpha y}{\text{Re}} - \alpha\psi = -\alpha[c_1 + c_2 y]\exp[\frac{-\alpha(x^2/2 - \alpha^2 \text{Re } x)}{\alpha^2 \text{Re} + 1}] .$$

…(35)

Case 3: $\beta < 0$

In this case, the characteristic roots in equation (25) are of the form

$$m = \mp i\sqrt{\beta}$$

…(36)

and equation (26) is replaced by

$$f(y) = c_1 \cos\sqrt{\beta}\, y + c_2 \sin\sqrt{\beta}\, y .$$

…(37)

The streamfunction, solution (27) is thus replaced by

$$\psi = \frac{y}{\text{Re}} + [c_1 \cos\sqrt{\beta}\, y + c_2 \sin\sqrt{\beta}\, y]\exp[\frac{-\alpha(x^2/2 - \alpha^2 \text{Re } x)}{\alpha^2 \text{Re} + 1}]$$

…(38)

with velocity components given by

$$u = \psi_y = \frac{1}{\text{Re}} + \sqrt{\beta}[-c_1 \sin\sqrt{\beta}\, y + c_2 \cos\sqrt{\beta}\, y]\exp[\frac{-\alpha(x^2/2 - \alpha^2 \text{Re } x)}{\alpha^2 \text{Re} + 1}]$$

…(39)

$$v = -\psi_x = -[c_1 \cos\sqrt{\beta}\, y + c_2 \sin\sqrt{\beta}\, y][\frac{-\alpha(x - \alpha^2 \text{Re})}{\alpha^2 \text{Re} + 1}]\exp[\frac{-\alpha(x^2/2 - \alpha^2 \text{Re } x)}{\alpha^2 \text{Re} + 1}]$$

…(40)

and Vorticity takes the form

$$\xi = \frac{\alpha y}{\text{Re}} - \alpha\psi = -\alpha[c_1 \cos\sqrt{\beta}\, y + c_2 \sin\sqrt{\beta}\, y]\exp[\frac{-\alpha(x^2/2 - \alpha^2 \text{Re } x)}{\alpha^2 \text{Re} + 1}] .$$

…(41)

IV. SUB-CLASSIFICATION OF FLOW

IV.1. Determining the value of α

Equation (14) gives the variable permeability as a function of x and shows its dependence on parameter α and on Reynolds number. For choices of α and Re, the value of dimensionless variability must be such that $0 < k(x) \le 1$. This implies that $\alpha > 0$. The value of α also affects the value of β, as given in its definition in equation (22) which also shows the influence of Re on β. Reynolds number must be greater than zero so that the permeability takes a positive value.

Equation (14) ties in together the values of permeability function at different values of x, and the values of parameter α for a given permeability value. We emphasize here that in obtaining the solutions discussed in this work, we assumed that $\text{Re} > 0$ and $x > 0$ so that the permeability has a positive numerical value, hence $\alpha > 0$. A minimum value of α is chosen so that the dimensionless permeability does not exceed 1. In what follows we will show that α must be greater than unity.

Limiting cases on the flow are as follows:

When $\text{Re} \to 0$, equation (22) shows that $\beta \to 2\alpha$, and equation (14) shows that $k \to 0$. This represents the limiting case for an impermeable solid.

Another limiting case is obtained when Re is large ($\text{Re} >> 1$), equation (22) shows that $\beta \to \alpha - \alpha^2$.

If $\beta = 0$ then $\alpha - \alpha^2 = 0$, or $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$, then $\xi = 0$ and the fluid is inviscid (which is not the case in the current flow problem). If $\alpha = 1$, we run into problems determining values of Reynolds number, as discussed in what follows.

The cases of $\beta = 0$, $\beta > 0$, and $\beta < 0$:

When $\beta = 0$, equation (22) reduces to:

$$(\alpha^4 - \alpha^5)\text{Re}^2 + 3\alpha^2 \text{Re} + 2 = 0 .$$

…(42)

If $\alpha = 1$ then Re is negative. A positive Re is therefore obtained when $\beta = 0$ and $\alpha > 0$ and $\alpha \neq 1$. Solution to equation (42) in terms of α is given by:

$$\text{Re} = \frac{3 \mp \sqrt{1 + 8\alpha}}{2(\alpha^3 - \alpha^2)} .$$

…(43)

Where we choose the sign of root that renders a positive value for Re, for a given $\alpha > 0, \alpha \neq 1$. This value of Re, for the choice of α represents the critical value that makes $\beta = 0$.

When $\beta > 0$ equation () reduces to:

$$\alpha^2(\alpha^3 - \alpha^2)\text{Re}^2 - 3\alpha^2 \text{Re} - 2 < 0$$

…(44)

and we must have $\alpha > 1$ in order to have a positive Reynolds number, given in terms of α as:

$$\text{Re} = \frac{3 + \sqrt{1 + 8\alpha}}{2(\alpha^3 - \alpha^2)} .$$

…(45)

A choice of positive Re that is less than or equal to the value calculated using equation (45) guarantees that $\beta > 0$.

When $\beta < 0$ equation (42) reduces to:

$$\alpha^2(\alpha^3 - \alpha^2)\text{Re}^2 - 3\alpha^2 \text{Re} - 2 > 0 .$$

…(46)

and we must have $0 < \alpha < 1$ in order to have a positive Reynolds number, given in terms of α as:

$$\text{Re} = \frac{3 - \sqrt{1 + 8a}}{2(a^3 - a^2)} .$$

…(47)

A choice of Re greater or equal to the value calculated using equation (47) guarantees that $\beta > 0$.

IV.2. Stagnation

The flow described by equations (27)-(30), (32)-(35) and (38)-(41) can be sub-classified according to the values of the arbitrary constants c_1 and c_2. In particular when $u = v = 0$ stagnation points in the flow occur.

Case 1: $\beta > 0$

Setting $u = v = 0$ in (28) and (29) gives the following values of (x, y) at stagnation:

$$x = \alpha^2 \mathrm{Re}$$
…(48)

$$y = \frac{1}{\sqrt{\beta}} \ln \left\{ \frac{-1}{2c_1 \mathrm{Re} \sqrt{\beta} \exp[\frac{\alpha^5 \mathrm{Re}^2/2}{\alpha^2 \mathrm{Re}+1}]} \right\} \mp \sqrt{\frac{c_1 + c_2 \left[4c_1 \mathrm{Re} \sqrt{\beta} \exp[\frac{\alpha^5 \mathrm{Re}^2/2}{\alpha^2 \mathrm{Re}+1}] \right]^2}{c_1 \left[4c_1 \mathrm{Re} \sqrt{\beta} \exp[\frac{\alpha^5 \mathrm{Re}^2/2}{\alpha^2 \mathrm{Re}+1}] \right]^2}} .$$
…(49)

In order to have a positive argument of the natural logarithmic function, we must choose $c_1 < 0$. The value of c_2 must be chosen such that

$$c_1 + c_2 \left[4c_1 \mathrm{Re} \sqrt{\beta} \exp[\frac{\alpha^5 \mathrm{Re}^2/2}{\alpha^2 \mathrm{Re}+1}] \right]^2 \le 0$$ or

$$c_2 \le \frac{-c_1}{\left[4c_1 \mathrm{Re} \sqrt{\beta} \exp[\frac{\alpha^5 \mathrm{Re}^2/2}{\alpha^2 \mathrm{Re}+1}] \right]^2} .$$
…(50)

We note that permeability to the fluid along the vertical lines (48) is given by

$$k = \frac{\alpha \mathrm{Re}}{x} = \frac{1}{\alpha} .$$
…(51)

When $c_2 \ge c_1$, solutions represent a reversing flow over a flat plate (situated to the right of the y-axis) with suction ($\psi - \frac{y}{\mathrm{Re}} > 0$) or blowing ($\psi - \frac{y}{\mathrm{Re}} < 0$). For non-negative c_1, suction occurs, and when $c_1 < 0$ blowing occurs. When $c_1 > c_2$, the flow is non-reversing with suction if $c_2 \ge 0$ or blowing if $c_2 < 0$.

Case 2: $\beta = 0$

Equations (33) and (34) give the following stagnation points:

$$y = -\frac{c_1}{c_2}$$
…(52)

$$x = \alpha \mathrm{Re} \mp \sqrt{\alpha^2 \mathrm{Re}^2 - \frac{2(\alpha^2 \mathrm{Re}+1)}{\alpha} \ln(-\frac{1}{c_2 \mathrm{Re}})} .$$
…(53)

For $\ln(-\frac{1}{c_2 \mathrm{Re}})$ to be defined, we much have $c_2 < 0$. In addition, we much have

$$\alpha^2 \mathrm{Re}^2 - \frac{2(\alpha^2 \mathrm{Re}+1)}{\alpha} \ln(-\frac{1}{c_2 \mathrm{Re}}) \ge 0$$
…(54)

Which is guaranteed by choosing $c_2 < 0$ such that

$$-c_2 \ge \frac{1}{\mathrm{Re} .\exp[\frac{\alpha^3 \mathrm{Re}^2}{2(\alpha^2 \mathrm{Re}+1)}]} .$$
…(55)

In order to have a positive value for permeability, we must have $x > 0$. Values of the parameters in (53) must be chosen to guarantee $x > 0$.

These solutions represent flow over a porous flat plate with suction or blowing.

Case 3: $\beta < 0$

Setting $u = 0$ and $v = 0$ in equations (39) and (40) results in

$$x = \alpha^2 \mathrm{Re} \mp \sqrt{\alpha^4 \mathrm{Re}^2 + (\alpha \mathrm{Re} + 1/\alpha)[\ln(\gamma \mathrm{Re})^2]}$$
…(56)

$$y = -\frac{1}{\sqrt{\beta}} \tan^{-1} \lambda$$
…(57)

where

$$\lambda = \frac{c_1}{c_2}$$
…(58)

and

$$\gamma = (c_1 \lambda - c_2) \frac{\sqrt{\beta}}{\sqrt{1+\lambda^2}} .$$
…(59)

It is clear that $c_1, c_2 \in \Re$; however the parameters in (56) must be chosen such that $x > 0$ so that permeability is positive. The obtained solution represents a flow field consisting of alternating vortices that are superposed on the main flow, perpendicular to their planes.

IV.3. Comparison with Constant Permeability Solutions

When the permeability is constant in the DLB equation, the following three exact solutions for the streamfunction have been obtained by Merabet et al. [17]. In both cases of constant or variable permeability, Reynolds number is connected to permeability. However, β is defined differently and its range is different in both flow types, while $\alpha = 1$ for constant permeability flow.

$$\xi = \alpha[Ry - \psi] \quad \ldots(60)$$

$$\beta = 1 - \frac{1}{R^2}\left\{\alpha - \frac{1}{k}\right\}^2 \quad \ldots(61)$$

$$R = \frac{1}{\mathrm{Re}}; \qquad \alpha = 1. \quad \ldots(62)$$

Case 1: $0 < \beta < 1$

$$\psi = Ry + c_1 \exp\left\{\frac{x}{R}\left[\frac{k-1}{k}\right] + \frac{y}{R}\sqrt{R^2 - \left\{\frac{k-1}{k}\right\}^2}\right\} + c_2 \exp\left\{\frac{x}{R}\left[\frac{k-1}{k}\right] - \frac{y}{R}\sqrt{R^2 - \left\{\frac{k-1}{k}\right\}^2}\right\} \quad \ldots(63)$$

$$\mathrm{Re} = \frac{1}{R} > \frac{k}{1-k}. \quad \ldots(64)$$

Case 2: $\beta = 0$

$$\psi = Ry + (d_1 + d_2 y)\exp\left\{\frac{x}{R}\left[\frac{k-1}{k}\right]\right\} \quad \ldots(65)$$

$$\mathrm{Re} = \frac{1}{R} = \frac{k}{1-k}. \quad \ldots(66)$$

Case 3: $\beta < 0$

$$\psi = Ry + \exp\left\{\frac{x}{R}\left[\frac{k-1}{k}\right]\right\} * \left[e_1 \cos\left(\frac{y}{R}\sqrt{\left\{\frac{k-1}{k}\right\}^2 - R^2}\right) + e_2 \sin\left(\frac{y}{R}\sqrt{\left\{\frac{k-1}{k}\right\}^2 - R^2}\right)\right] \quad \ldots(67)$$

$$\mathrm{Re} = \frac{1}{R} < \frac{k}{1-k}. \quad \ldots(68)$$

V. CONCLUSION

In this work we have obtained three exact solutions to the Darcy-Lapwood-Brinkman equation with variable permeability. Permeability has been defined as a function of one space dimension, and vorticity is prescribed as a function of the streamfunction of the flow. Ranges of parameters have been determined, and comparison is made with the solutions obtained for the constant permeability case. Solutions obtained represent fields of flow over a flat plate with blowing or suction.

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